CHAPTER 2

FIRST ORDER DIFFERENTIAL EQUATIONS

In this section we discuss how to solve a first order differential equation in the form

$F(x, y,\ y') = 0 \qquad or \qquad y' = f(x, y)$

and the general solution can be written in the form $f(x, y, c) = 0$, where c is an arbitrary constant.

■1. SEPARATION OF VARIABLES►

We use this method if we can write the given differential equation in the form

$$y' = P(x)Q(y)$$

or in the form

$$P(x)dx + Q(y)dy = 0.$$

By integrate the above equation we get the general solution of the differential

EXAMPLE 1.1►

Find the general solution of the D.E.

$$y' = 3x^2 + 1$$

SOLUTION

By separate the variables

$$dy = (3x^2 + 1)dx$$

By integrate the both sides

$$\int dy = \int (3x^2 + 1)dx$$

Then the general solution is

$$y = x^3 + x + c$$

EXAMPLE 1.2►

Find the general solution of the D.E.

$$(1 + x^2)y' = 1 + y^2$$

SOLUTION

By separate the variables

$$\frac{dy}{1 + y^2} = \frac{dx}{1 + x^2}$$

By integrate the both sides

$$\int \frac{dy}{1 + y^2} = \int \frac{dx}{1 + x^2}$$

Then the general solution is

$$tan^{-1}x = tan^{-1}y + c$$

EXAMPLE 1.3►

Find the general solution of the D.E.

$$\frac{dy}{dx} = \frac{x + \cos x}{y^2}$$

SOLUTION

Separation of variables yields

$$y^2 dy = (x + \cos x)dx$$

By integrate the both sides

$$\int y^2 dy = \int (x + \cos x) dx$$

Then the general solution is

$$y^3 = \frac{3}{2}x^2 + 3\sin x + c$$

EXAMPLE 1.4►

Find the general solution of the D.E.

$$\mathrm{x}dx + \mathrm{y}dy = 0 \quad , \quad \mathrm{y}(\sqrt{2}) = \sqrt{2}$$

SOLUTION

By integrate the both sides

$$\int x dx + \int y dy = c$$

Then the general solution is

$$x^2 + y^2 = c$$

To determine c: since $x = \sqrt{2}\ at\ y = \sqrt{2}$,then the particular solution is $x^2 + y^2 = 4$

EXAMPLE 1.5►

Find the general solution of the D.E.

$$\mathrm{y}' + 2\mathrm{x}\sqrt{1 - \mathrm{y}^2} = 0$$

SOLUTION

By dividing over $\sqrt{1 - y^2}$ and separate the variables we get

$$\frac{dy}{\sqrt{1 - y^2}} + 2xdx = 0$$

By integrate the both sides

$$\int \frac{dy}{\sqrt{1-y^2}} = -2\int x dx$$

we have $sin^{-1}y = c - x^2$

Then the general solution is

$$y = \sin(c - x^2)$$

EXAMPLE 1.6►

Find the general solution of the D.E.

$$yy' + x^3(y^2 - 1) = 0$$

SOLUTION

By separate the variables

$$\frac{ydy}{y^2 - 1} = x^3 dx$$

By integrate the both sides

$$\frac{1}{2}\int \frac{2ydy}{y^2 - 1} = -\int x^3 dx$$

Then the general solution is

$$\frac{1}{2}\ln(y^2 - 1) = \frac{1}{4}x^4 + c$$

EXAMPLE 1.7►

Find the particular solution of the D.E.

$$yy' + x^2 e^{x^3 - y^2} = 0 \quad \text{where} \quad y(0) = 0$$

SOLUTION

By separate the variables

$$x^2 e^{x^3} dx + y e^{y^2} dy =$$

By integrate the both sides

$$\int x^2 e^{x^3} dx + \int y e^{y^2} dy = 0$$

Then the general solution is

$$\frac{1}{3} e^{x^3} + \frac{1}{2} e^{y^2} = c$$

From the initial condition $y(0) = 0$ the particular solution is

$$2e^{x^3} + 3e^{y^2} = 5$$

EXAMPLE 1.8►

Find the general solution of the D.E.

$$sec^2 x \tan y dx + sec^2 y \tan x \, dy = 0$$

SOLUTION

By separate the variables

$$\frac{sec^2 x}{\tan x} dx + \frac{sec^2 y}{\tan y} dy = 0$$

By integrate the both sides

$$\int \frac{sec^2 x}{\tan x} dx + \int \frac{sec^2 y}{\tan y} dy = 0$$

Then the general solution is

$$ln|\tan x| + ln|\tan y| = \ln c$$

$$\tan x \tan y = c$$

EXAMPLE 1.9►

Find the general solution of the D.E.

$$y' + e^x y = e^x y^2$$

SOLUTION

Write the equation in the form

$$\frac{dy}{dx} = e^x y(y - 1)$$

Separation of variables yields

$$\frac{dy}{y(y-1)} = e^x dx$$

By integrate using the partial fractions we have

$$\ln|y-1| - \ln|y| = e^x + c$$

Then the general solution is

$$\frac{\ln|y-1|}{\ln|y|} = e^x + c$$

EXAMPLE 1.10►

Find the general solution of the D.E.

$$(1+y^2)dx + (1+x^2)dy = 0$$

SOLUTION

By separate the variables

$$\frac{dx}{(1+x^2)} + \frac{dy}{(1+y^2)} = 0$$

By integrate the both sides

$$\int \frac{dx}{(1+x^2)} + \int \frac{dy}{(1+y^2)} = 0$$

Then the general solution is

$$tan^{-1}x + tan^{-1}y = c$$

EXERCISES 1

Find the general solution of the following D.E. and if the initial condition given investigate the particular solution

(1) $y' = 2xy$

(2) $x^2 y dy = e^y dx$

(3) $\frac{y^2}{x}\frac{dy}{dx} = 1 + x^2$

(4) $\frac{dy}{dx} = \cos x\, e^{-y}$

(5) $x\sqrt{1+y^2} + y\sqrt{1+x^2}y' = 0$

(6) $2xyy' = 1 + x$

(7) $\frac{dy}{dx} = x^2(1+y^2)$

(8) $y' = \frac{y}{x}$, $y(1) = 2$

(9) $(1+x^2)y' = x^4 y^4$

(10) $y' = y \sec x$

(11) $e^{x+y}y' = 2x$

(12) $y' = \frac{yx^2 - 2xy + y}{y-3}$

(13) $y' = e^{x-y} + x^2 e^{-y}$

(14) $2xyy' = 1 + y^2$, $y(2) = 3$

(15) $y' = e^{2x-3y}$, $y(0) = 0$

(16) $xy' - y = xy$

(17) $xy' = \cot y$, $y(1) = 0$

(18) $y' = \frac{a^2+y^2}{b^2+x^2}$, a, b consts.

(19) $(a^2 + x^2)dy - ydx = 0$

(20) $e^y y' = 3(1+e^y)x^2$

(21) $(1+x)ydx + (1-y)xdy = 0$

(22) $\sin x \cos y\, dx - \cos x \sin y\, dy = 0$

(23) $3e^x \tan y + y'(1+e^x)sec^2 y = 0$, $y(0) = \frac{\pi}{4}$

(24) $x(x+1)dy = (y+1)(y+2)dx$

■2. HOMOGENEOUS D.E.►

In this section we will discuss how to solve homogeneous differential equations of first order and differential equations can be transformed to homogeneous D.E.

DEFINITION►

The function $f(x,y)$ is homogeneous of degree n if

$$f(\lambda x,\lambda y)=\lambda^n f(x,y) \quad where\ \lambda\ is\ a\ parameter$$

EXAMPLE 2.1►

Discuss the homogeneity of the following functions:

$(a)\ \ f(x,y,z)=\sqrt{x^3z-y^2z^2}$

$(b)\ f(x,y)=\dfrac{x^2+y^2}{xy}$

$(c)\ \ f(x,y)=x\tan\left(\dfrac{y}{x}\right)+y\cos\left(\dfrac{y}{x}\right)$

$(d)\ \ f(x,y)=x^2-y^2+x^3y^3$

SOLUTION

$$(a)\ f(\lambda x,\lambda y,\lambda z)=\sqrt{(\lambda x)^3(\lambda z)-(\lambda y)^2(\lambda z)^2}$$

$$=\sqrt{\lambda^4(x^3z-y^2z^2)}$$

$$=\lambda^2\sqrt{x^3z-y^2z^2}=\lambda^2 f(x,y)$$

i.e. the function is homogeneous of degree 2.

$$(b)\ f(\lambda x,\lambda y,\lambda z)=\frac{(\lambda x)^2+(\lambda y)^2}{(\lambda x)(\lambda y)}=\frac{\lambda^2(x^2+y^2)}{\lambda^2 xy}$$

$$=\frac{(x^2+y^2)}{xy}=f(x,y,z)$$

i.e. the function is homogeneous of degree 0.

$$(c)\ f(\lambda x,\lambda y)=\lambda x\tan\left(\frac{\lambda y}{\lambda x}\right)+\lambda y\cos\left(\frac{\lambda y}{\lambda x}\right)$$

$$=\lambda\left[x\tan\left(\frac{y}{x}\right)+y\cos\left(\frac{y}{x}\right)\right]=\lambda f(x,y)$$

i.e. the function is homogeneous of degree 1.

$$(d)\ f(\lambda x,\lambda y)=\lambda^2x^2-\lambda^2y^2+\lambda^6x^3y^3$$

$$=\lambda^2\left(x^2-y^2+\lambda^4x^3y^3\right)\neq\lambda^2f(x,y)$$

i.e. the function is non-homogeneous since its value depends on λ.

Let

$$y'=f(x,y)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

homogeneous differential equation

Then it need not be separable. However a homogeneous differential equation is always transformed into separable equation by using the substitution

$$v=\frac{y}{x}$$

Substitution into differential equation (1) by

$$y=vx\ \rightarrow y'=v+xv'$$

We have $v+xv'=f(v)$

And by separate the variables we get

$$\frac{dv}{f(v)-v}=\frac{dx}{x}$$

Therefore, equation (1) is transformed into separable D.E.

EXAMPLE 2.2►

Find the general solution of the D.E

$$y' = \frac{y}{x} + \frac{x}{y}$$

SOLUTION
The D.E. is homogeneous, so we put

$$y = xv \quad \rightarrow y' = xv' + v$$

$$xv' + v = v + \frac{1}{v}$$

$$x\frac{dv}{dx} = \frac{1}{v}$$

This separate equation in terms of x, v has differential form

$$\int v\, dv = \int \frac{dx}{x} + c$$

Integration the both sides, we obtain

$$\frac{1}{2}v^2 = \ln|x| + c$$

The general solution can be obtained by substituting $v = \frac{y}{x}$

$$y^2 = x^2(lnx^2 + 2c)$$

EXAMPLE 2.3►

Find the general solution of the D.E

$$xy' = xe^{\frac{y}{x}} + y$$

SOLUTION
The D.E. is not separable, so we write it as

$$y' = e^{\frac{y}{x}} + \frac{y}{x}$$

Which recognize as being homogeneous. So by putting

$$y = xv \quad \rightarrow y' = xv' + v$$

$$xv' + v = e^v + v$$

$$x\frac{dv}{dx} = e^{v}$$

This separate equation in terms of x, v has differential form

$$\int e^{-v}\, dv = \int \frac{dx}{x} + c$$

Integration the both sides, we obtain

$$-e^{-v} = \ln|x| + c$$

The general solution can be obtained by substituting $v = \frac{y}{x}$

$$-e^{\frac{-y}{x}} = \ln|x| + c$$

EXAMPLE 2.4►

Find the general solution of the D.E

$$(x - y)y' = y$$

SOLUTION

Dividing this equation by x (assuming that $x \neq 0$), we have

$$(1 - \frac{y}{x})y' = \frac{y}{x}$$

Let $y = xv \quad \rightarrow y' = xv' + v$

$$xv' + v = \frac{v}{1 - v}$$

$$x\frac{dv}{dx} = \frac{v^2}{1 - v}$$

In differential form, this equation can be written

$$\frac{1 - v}{v^2} = \frac{dx}{x}$$

Integration the both sides, we get

$$-\frac{1}{v} - \ln|v| = \ln|x| + c$$

The general solution can be obtained by substituting $v = \frac{y}{x}$

$$-\frac{x}{y} - \ln\left|\frac{y}{x}\right| = \ln|x| + c$$

EXAMPLE 2.5►

Solve the differential equation

$$xy' = y + \sqrt{x^2 - y^2}$$

SOLUTION

Dividing this equation by x (assuming that $x \neq 0$), we have

$$y' = \frac{y}{x} + \left(1 - \frac{y^2}{x^2}\right)^{\frac{1}{2}}$$

This is a homogeneous D.E.

Let $y = xv \quad \rightarrow y' = xv' + v$

$$xv' + v = v + (1 - v^2)^{\frac{1}{2}}$$

$$x\frac{dv}{dx} = (1 - v^2)^{\frac{1}{2}}$$

$$\frac{dv}{\sqrt{1 - v^2}} = \frac{dx}{x}$$

Integration the both sides, we get

$$sin^{-1}v = \ln|x| + \ln c$$

Substituting $v = \frac{y}{x}$

$$y = x\ sin\ (\ln cx)$$

EXAMPLE 2.6►

Solve the differential equation

$$xy' = y + \sqrt{x^2 + y^2}$$

SOLUTION

Dividing this equation by x (assuming that $x \neq 0$), we have

$$y' = \frac{y}{x} + \left(1 + \left(\frac{y}{x}\right)^2\right)^{\frac{1}{2}}$$

This is a homogeneous D.E.

Let $y = xv \quad \rightarrow y' = xv' + v$

$$xv' + v = v + (1 + v^2)^{\frac{1}{2}}$$

$$\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integration the both sides and substituting $v = \frac{y}{x}$

$$sinh^{-1}\left(\frac{y}{x}\right) = \ln|cx|$$

EXAMPLE 2.7►

Solve the differential equation

$$(xy + x^2)dy - y^2dx = 0$$

SOLUTION

The D.E. is homogeneous, so let

$$y = xv \quad \rightarrow y' = xv' + v$$

$$xv' + v = \frac{v^2}{v+1}$$

$$x\frac{dv}{dx} = \frac{-v}{v+1}$$

By separate and integrate the both sides, we have

$$-v - \ln|v| = \ln|x| + \ln c$$

Substituting $v = \frac{y}{x}$ the general solution is

$$cy = e^{\frac{-y}{x}}$$

EXAMPLE 2.8►

Find the general solution of the D.E

$$y' = \frac{x+y}{x-y}$$

SOLUTION

The D.E. is homogeneous of zero degree so we can rewrite it as

$$y' = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

Let $y = xv \quad \rightarrow y' = xv' + v$

$$xv' + v = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

By separate and integrate the both sides, we have

$$\int \frac{dv}{1+v^2} - \int \frac{vdv}{1+v^2} = \int \frac{dx}{x} + c$$

$$tan^{-1}v - \frac{1}{2}\ln(1+v^2) = \ln|x| + \ln c$$

The general solution can be obtained by substituting $v = \frac{y}{x}$

$$tan^{-1}\frac{y}{x} = \ln c\sqrt{x^2+y^2}$$

EXAMPLE 2.9►

Find the general solution of the D.E

$$2e^{y/x}\left(1-\frac{y}{x}\right)dx + \left(1+2e^{y/x}\right)dy = 0$$

SOLUTION

The D.E. is homogeneous of zero degree so

Let $y = xv \quad \rightarrow dy = xdv + vdx$

$$2e^v(1-v)dx + (1+2e^v)(xdv+vdx) = 0$$

$$(2e^v + v)dx + x(1+2e^v)dv = 0$$

By separate and integrate the both sides, we have

$$\int \frac{2e^v + 1}{2e^v + v} dv + \int \frac{dx}{x} = 0$$

$$\ln|2e^v + v| + \ln|x| = \ln c$$

The general solution can be obtained by substituting $v = \frac{y}{x}$

$$y + 2xe^{\frac{y}{x}} = c$$

EXAMPLE 2.10►

Find the general solution of the D.E

$$x^2y' = x^2 + xy + y^2$$

SOLUTION

We can rewrite the D.E. as

$$y' = \frac{x^2 + xy + y^2}{x^2}$$

The D.E. is homogeneous, so we put

$$y = xv \quad \rightarrow y' = xv' + v$$

$$xv' + v = \frac{1 + v + v^2}{1}$$

By separate and integrate the both sides, we have

$$\int \frac{dv}{1 + v^2} = \int \frac{dx}{x} + \ln c$$

$$tan^{-1}v = \ln|x| + \ln c$$

The general solution can be obtained by substituting $v = \frac{y}{x}$

$$y = x\, tan(\ln xc)$$

EXERCISES 2

Find the general solution of the following D.E. and if the initial condition given investigate the particular solution

(1) $xy' - y = \sqrt{xy}$	*(2)* $(x+y)dx + xdy = 0$
(3) $xdy - ydx - \sqrt{x^2 - y^2}dx = 0$	*(4)* $2x^2y' + 5x^2 + y^2 = 0$
(5) $(3x - y)y' = 3y - x$	*(6)* $y' = \frac{y}{x} + e^{y/x}$
(7) $xydx - (x - 2y)^2dy = 0$	*(8)* $(x^2 + y^2)y' = xy$
(9) $3x(y + x)y' = 2(2y^2 + 2xy + x^2)$	*(10)* $xy' = y + xe^{y/x}$
(11) $(x^2 + y^2)dx + xydy = 0, y(1) = -1$	*(12)* $y' = \frac{y}{x} + sin\frac{y}{x}$
(13) $xy' = y + \sqrt{4x^2 + y^2},\ y(1) = 0$	*(14)* $(x + y)y' = y$
(15) $\left(xsin\ \frac{y}{x} - ycos\ \frac{y}{x}\right)dx + xcos\ \frac{y}{x}dy = 0$	*(16)* $y^2dx = x(2y - x)dy$
(17) $(x^3 + y^3)dx + 3xy^2dy = 0$	*(18)* $(x^2 - y^2)dx = xydy$

■3.DIFFERENTIAL EQUATIONS CAN BE TRANSFORMED INTO HOMOGENEOUS D.E.►

Consider the D.E. in the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \dots\dots\dots\dots\dots (1)$$

Where the numerator and denominator represent two straight lines

We have three cases:

First case

If $c_1 = c_2 = 0$ then the D.E. (1) is homogeneous D.E. can be solved as we explain in the previous section.

Second case

If the two straight lines are intersecting,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \quad or \quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Let we assume the intersection point is *(h, k),* then put

$$x = u + h\,, \qquad y = v + k \;\; \Longmapsto \;\; dx = du, \qquad dy = dv$$

Substituting in the D.E. (1) we get

$$\frac{dv}{du} = \frac{a_1 u + b_1 v + (a_1 h + b_1 k + c_1)}{a_2 u + b_2 v + (a_2 h + b_2 k + c_2)} \ldots\ldots\ldots\ldots (2)$$

Since *(h, k)* is the intersection point of the two straight lines, then

$$a_1 h + b_1 k + c_1 = 0 \;\; \& \;\; a_2 h + b_2 k + c_2 = 0$$

and the D.E. (2) become in the form

$$\frac{dv}{du} = \frac{a_1 u + b_1 v}{a_2 u + b_2 v},$$

which is a homogeneous D.E.

EXAMPLE 3.1►

Find the general solution of the D.E

$$y' = \frac{x + y - 1}{y - x - 3}$$

SOLUTION

It's clear that the two lines $x + y - 1 = 0, y - x - 3 = 0$ are intersecting. So, solving the two equations to determine the intersection *(h, k)* we get

$$(h, k) = (-1, 2)$$

Put $x = u - 1, \quad y = v + 2$, the given D.E will be

$$\frac{dv}{du} = \frac{u + v}{v - u}$$

Which is homogeneous D.E., so we put

$$v = uz \quad \rightarrow v' = uz' + z$$

$$-uz' = \frac{z^2 - 2z - 1}{z - 1}$$

By separate and integrate the both sides, we have

$$\int \frac{z-1}{z^2 - 2z - 1} dz = -\int \frac{du}{u}$$

$$\frac{1}{2}\ln|z^2 - 2z - 1| = -\ln|u| + \ln c$$

$$u^2(z^2 - 2z - 1) = C$$

By substituting $z = \frac{v}{u}$ we obtain

$$u^2\left[\left(\frac{v}{u}\right)^2 - 2\left(\frac{v}{u}\right) - 1\right] = C$$

$$v^2 - 2uv - u^2 = C$$

Substituting $u = x + 1,\ \ v = y - 2$ the general solution is

$$(y-2)^2 - 2(x+1)(y-2) - (x+1)^2 = C$$

EXAMPLE 3.2►

Find the general solution of the D.E

$$y' = \frac{2x + y - 3}{x + y - 2}$$

SOLUTION

It's clear that the two lines $2x + y - 3 = 0$, $x + y - 2 = 0$ are intersecting and by solving the two equations to determine the intersection *(h, k)* we get

$$(h, k) = (1,1)$$

Put $x = u + 1,\ \ y = v + 1$, the given D.E will be

$$\frac{dv}{du} = \frac{2u+v}{u+v}$$

Which is homogeneous D.E., so we put

$$v = uz \quad \rightarrow v' = uz' + z$$

$$uz' + z = \frac{2+z}{1+z}$$

$$uz' = \frac{2+z}{1+z} - z = \frac{2-z^2}{1+z}$$

By separate and integrate the both sides, we have

$$\int \frac{1+z}{2-z^2}\,dz = \int \frac{du}{u}$$

$$\int \frac{1}{2-z^2}\,dz + \int \frac{z}{2-z^2}\,dz = \ln|u| + c$$

Using partial fraction integral we have

$$\frac{\sqrt{2}}{4} ln \left|\frac{\sqrt{2}+z}{\sqrt{2}-z}\right| - \frac{1}{2}\ln|2-z^2| = \ln|u| + c$$

By substituting $z = \frac{v}{u}$ we obtain

$$\frac{\sqrt{2}}{4} ln \left|\frac{\sqrt{2}+\frac{v}{u}}{\sqrt{2}-\frac{v}{u}}\right| - \frac{1}{2}\ln\left|2-\left(\frac{v}{u}\right)^2\right| = \ln|u| + c$$

Substituting $u = x-1,\ \ v = y-1$ the general solution is

$$\frac{\sqrt{2}}{4} ln \left|\frac{\sqrt{2}(x-1)+y-1}{\sqrt{2}(x-1)-y+1}\right| - \frac{1}{2}\ln\left|2-\left(\frac{y-1}{x-1}\right)^2\right| = \ln|x-1| + c$$

EXAMPLE 3.3►

Find the general solution of the D.E

$$y' = \frac{x+y-3}{x-y-1}$$

SOLUTION

The two lines are intersecting and by solving the two equations to determine the intersection *(h, k)* we get

$$(h, k) = (2,1)$$

Put $x = u + 2, \quad y = v + 1$, the given D.E will be

$$\frac{dv}{du} = \frac{u + v}{u - v}$$

Which is homogeneous D.E., so we put

$$v = uz \quad \rightarrow v' = uz' + z$$

$$uz' + z = \frac{1 + z}{1 - z}$$

By separate and integrate the both sides, we have

$$\int \frac{1 - z}{1 + z^2} dz = \int \frac{du}{u}$$

$$tan^{-1}z - \frac{1}{2}\ln|1 + z^2| = \ln|u| + c$$

By substituting $z = \frac{v}{u}$ we obtain

$$tan^{-1}\left(\frac{v}{u}\right) - \frac{1}{2}\ln\left|1 + \left(\frac{v}{u}\right)^2\right| = \ln|u| + c$$

Substituting $u = x - 2, \quad v = y - 1$ the general solution is

$$tan^{-1}\left(\frac{y - 1}{x - 2}\right) - \frac{1}{2}\ln\left|1 + \left(\frac{y - 1}{x - 2}\right)^2\right| = \ln|x - 2| + c$$

EXAMPLE 3.4►

Find the general solution of the D.E

$$y' = \frac{x + 3y - 2}{3x + y + 2}$$

SOLUTION

One can notice that the two lines are intersection and by solving the two equations the intersection point is

$$(h, k) = (-1,1)$$

Put $x = u - 1, \quad y = v + 1$, the given D.E will be

$$\frac{dv}{du} = \frac{u + 3v}{3u + v}$$

Which is homogeneous D.E., so we put

$$v = uz \quad \rightarrow v' = uz' + z$$

$$uz' + z = \frac{1 + 3z}{3 + z}$$

By separate and integrate the both sides, we have

$$\int \frac{3 + z}{1 - z^2} dz = \int \frac{du}{u}$$

$$\int \frac{du}{u} + \int \left(\frac{2}{z - 1} - \frac{1}{z + 1}\right) dz = \ln c$$

$$\ln|u| + 2\ln|z - 1| - \ln|z + 1| = \ln c$$

By substituting $z = \frac{v}{u} = \frac{y-1}{x+1}$, we obtain

$$\ln|x + 1| + 2\ln\left|\frac{y - 1}{x + 1} - 1\right| - \ln\left|\frac{y - 1}{x + 1} + 1\right| = \ln c$$

$$\ln\left|(x + 1)\left(\frac{y - x - 2}{x + 1}\right)^2\right| = \ln c \left|\frac{y + x}{x + 1}\right|$$

The general solution is

$$(x - y + 2)^2 = c(x + y)$$

Third case

If the two straight lines are parallel:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \quad or \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

In this case we using the substitution

$$z = a_1 x + b_1 y \qquad or \qquad z = a_2 x + b_2 y\,,$$

which more simply. In this case the D.E. transformed into separable equation

EXAMPLE 3.5►

Find the general solution of the D.E

$$y' = \frac{x + 2y + 3}{2x + 6y + 7}$$

SOLUTION

The two straight lines are parallel, so we put

$$z = x + 2y \quad \Longmapsto \frac{dz}{dx} = 1 + 2\frac{dy}{dx}$$

$$\frac{dz}{dx} = 1 + 2\left(\frac{z + 3}{2z + 7}\right)$$

$$\frac{dz}{dx} = \frac{4z + 13}{2z + 7}$$

$$\frac{2(2z + 7) - 1}{2z + 7}dz = dx$$

$$\int 2dz - \int \frac{dz}{2z + 7} = \int dx$$

$$2z - \frac{1}{2}\ln|2z + 7| = x + c$$

$$2x + 4y - \frac{1}{2}\ln|2x + 4y + 7| = x + c$$

EXAMPLE 3.6►

Find the general solution of the D.E

$$y' = \frac{x + y - 5}{x + y + 1}$$

SOLUTION

We notice that the two straight lines are parallel, so let

$$z = x + y \quad \Longmapsto \frac{dy}{dx} = \frac{dz}{dx} - 1$$

Substitution in the D.E.

$$\frac{dz}{dx} - 1 = \frac{z - 5}{z + 1}$$

$$\frac{dz}{dx} = \frac{2z - 4}{z + 1}$$

Separate and integral we find

$$\int \frac{z + 1}{2z - 4} dz = \int dx$$

$$\frac{1}{2} z + \frac{3}{2} \ln|z - 2| = x + c$$

Substituting $z = x + y$ the general solution is

$$\frac{1}{2}(x + y) + \frac{3}{2} \ln|x + y - 2| = x + c$$

EXAMPLE 3.7►

Find the general solution of the D.E

$$y' = \frac{2x + y - 1}{4x + 2y + 5}$$

SOLUTION

We notice that the two straight lines are parallel, so let

$$z = 2x + y \quad \mapsto \frac{dy}{dx} = \frac{dz}{dx} - 2$$

Substitution in the D.E.

$$\frac{dz}{dx} - 2 = \frac{z-1}{2z+5}$$

$$\frac{dz}{dx} = \frac{5z+9}{2z+5}$$

Separate and integral we find

$$\int \frac{2z+5}{5z+9} dz = \int dx$$

$$\frac{2}{5}z + \frac{7}{25}\ln|5z+9| = x + c$$

Substituting $z = 2x + y$ the general solution is

$$\frac{2}{5}(2x+y) + \frac{7}{25}\ln|10x+5y+9| = x + c$$

EXAMPLE 3.8►

Find the general solution of the D.E

$$y' = \frac{2x+y}{4x+2y+1}$$

SOLUTION

We notice that the two straight lines are parallel, so let

$$z = 2x + y \quad \mapsto \frac{dy}{dx} = \frac{dz}{dx} - 2$$

Substitution in the D.E.

$$\frac{dz}{dx} - 2 = \frac{z}{2z+1}$$

$$\frac{dz}{dx} = \frac{5z+2}{2z+1}$$

Separate and integral we find

$$\int \frac{2z+1}{5z+2}dz = \int dx$$

$$\frac{1}{5}\int \left(2+\frac{1}{5z+2}\right)dz = \int dx$$

$$\frac{1}{5}\left(2z+\frac{1}{5}\ln|5z+2|\right) = x+c$$

Substituting $z = 2x+y$ the general solution is

$$(4x+2y)+\frac{1}{5}\ln|10x+5y+2| = 5x+c$$

EXERCISES 3

Find the general solution of the following D.E.

(1) $\frac{dy}{dx} = \frac{x+7y+2}{3x+5y+6}$	(2) $\frac{dy}{dx} = \frac{x+y+2}{x-y-4}$
(3) $\frac{dy}{dx} = \frac{x-y-1}{x-y-5}$	(4) $\frac{dy}{dx} = \frac{x-2y+9}{3x-6y+19}$
(5) $\frac{dy}{dx} = \frac{-2x+2y}{y-1}$	(6) $\frac{dy}{dx} = \frac{x-3y+2}{3x+9y-12}$
(7) $\frac{dy}{dx} = \frac{-x+y}{-x+y+1}$	(8) $\frac{dy}{dx} = \frac{x+y-3}{x+y+5}$
(9) $\frac{dy}{dx} = \frac{x+4y-2}{4x-y+9}$	(10) $\frac{dy}{dx} = \frac{x+2y+1}{2x+4y+3}$
(11) $\frac{dy}{dx} = \frac{2x+y-2}{y-x-2}$	(12) $\frac{dy}{dx} = \frac{7x-3y-7}{7y-3x+3}$
(13) $\frac{dy}{dx} = \frac{2x+2y+1}{x+y-1}$	(14) $\frac{dy}{dx} = \frac{x+y-1}{2x+2y-3}$
(15) $\frac{dy}{dx} = \frac{3x+4y+1}{4x+5y+4}$	(16) $\frac{dy}{dx} = \frac{6x-3y+2}{2x-y-1}$
(17) $\frac{dy}{dx} = \frac{x-y-1}{x-y-2}$	(18) $\frac{dy}{dx} = \frac{x-y}{3x-3y+1}$

■4.EXACT DIFFERENTIAL EQUATIONS.►

Consider a first order differential equation in the form

$$P(x,y)dx + Q(x,y)dy = 0 \dots\dots\dots\dots\dots\dots (1)$$

The D.E. (1) is said to be **exact D.E.** if we can find a function *F(x, y)* whose total differential,

$$dF = P(x,y)dx + Q(x,y)dy,$$

In this case the general solution of Eq. (1) is

$$F(x,y) = c$$

Theorem ►

If $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous functions then the sufficient condition to become the D.E. (1) is exact D.E. is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Proof ►

Let $P(x,y)dx + Q(x,y)dy = 0$ is exact then there exist a function $F(x,y)$ such that

$$dF = P(x,y)dx + Q(x,y)dy$$

But from the differentiation for a function of two variables we find

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

By comparing the above two equations we find

$$\frac{\partial F}{\partial x} = P \qquad \frac{\partial F}{\partial y} = Q$$

Using the partial derivative we have

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial P}{\partial y} \quad , \qquad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

Since the function $F(x,y)$ is continuous then

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

So we can say that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

How we solve the exact D.E. ►

We have many methods to solve the exact differential equation.

The best and easiest of these methods that we find

$$(i) \int P(x,y)dx \qquad\qquad (ii) \int Q(x,y)dy$$

And the general solution be

all terms in (i)+ all terms in (ii) and doesn't appear in (i)=const.

EXAMPLE 4.1►

Find the general solution of the D.E

$$(2xy^3 + 2)dx + (3x^2y^2 + e^y)dy = 0$$

SOLUTION

Let $P = 2xy^3 + 2$, $Q = 3x^2y^2 + e^y$

$$\frac{\partial P}{\partial y} = 6xy^2 \quad , \quad \frac{\partial Q}{\partial x} = 6xy^2$$

Since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Then the D.E. is exact, so

$$(i) \int (2xy^3 + 2)dx = x^2y^3 + 2x$$

$$(ii)\int(3x^2y^2+e^y)dy=x^2y^3+e^y$$

And the general solution is

$$x^2y^3+2x+e^y=c$$

EXAMPLE 4.2►

Find the general solution of the D.E

$$[\cos x.\tan y+\cos(x+y)]dx+[\sin x.\sec^2 y+\cos(x+y)]dy=0$$

SOLUTION

By comparing the given equation and standard form we get

$$P=\cos x.\tan y+\cos(x+y)\quad,\quad Q=\sin x.\sec^2 y+\cos(x+y)$$

$$\frac{\partial P}{\partial y}=\cos x.\sec^2 y-\sin(x+y)\,,$$

$$\frac{\partial Q}{\partial x}=\cos x.\sec^2 y-\sin(x+y)$$

Since

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

Then the D.E. is exact, so

$$(i)\int[\cos x.\tan y+\cos(x+y)]dx=\sin x\tan y+\sin(x+y)$$

$$(ii)\int[\sin x.\sec^2 y+\cos(x+y)]dy=\sin x\tan y+\sin(x+y)$$

And the general solution is

$$\sin x\tan y+\sin(x+y)=c$$

EXAMPLE 4.3►

Find the general solution of the D.E

$$\frac{dy}{dx}=\frac{2x^3+3y}{1-3x-y}$$

SOLUTION

$$(2x^3 + 3y)dx + (3x + y - 1)dy = 0$$

Let $P = 2x^3 + 3y$, $Q = 3x + y - 1$

$$\frac{\partial P}{\partial y} = 3 \quad , \quad \frac{\partial Q}{\partial x} = 3$$

Since

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Then the D.E. is exact, so

$$(i) \int (2x^3 + 3y)dx = \frac{1}{2}x^4 + 3xy$$

$$(ii) \int (3x + y - 1)dy = 3xy + \frac{1}{2}y^2 - y$$

And the general solution is

$$x^4 + 6xy + y^2 - 2y = c$$

EXAMPLE 4.4►

Find the general solution of the D.E

$$\left(2xe^{x^2}y + 3x^2\right)dx + \left(e^{x^2} - 2y\right)dy = 0$$

SOLUTION

Let $P = 2xe^{x^2}y + 3x^2$, $Q = e^{x^2} - 2y$

$$\frac{\partial P}{\partial y} = 2xe^{x^2} \quad , \quad \frac{\partial Q}{\partial x} = 2xe^{x^2}$$

Then the D.E. is exact, so we find

$$(i) \int \left(2xe^{x^2}y + 3x^2\right)dx = e^{x^2}y + x^3$$

$$(ii) \int \left(e^{x^2} - 2y\right)dy = e^{x^2}y - y^2$$

is

$$e^{x^2}y + x^3 - y^2 = c$$

EXAMPLE 4.5►

Find the general solution of the D.E

$$\frac{2x(y-1)}{1+x^2}dx + \ln(1+x^2)\,dy = 0$$

SOLUTION

Let $P = \frac{2x(y-1)}{1+x^2}$, $Q = \ln(1+x^2)$

$$\frac{\partial P}{\partial y} = \frac{2x}{1+x^2} \quad , \quad \frac{\partial Q}{\partial x} = \frac{2x}{1+x^2}$$

Then the D.E. is exact, so we find

$$(i)\int \frac{2x(y-1)}{1+x^2}dx = (y-1)\ln(1+x^2)$$

$$(ii)\int \ln(1+x^2)\,dy = \mathrm{y}\ln(1+x^2)$$

And the general solution is

$$(y-1)\ln(1+x^2) = c$$

EXERCISES 4

Find the general solution of the following D.E.

(1) $2xydx + (x^2+y^2)dy = 0$

(2) $e^{-y}dx - (2y + xe^{-y})dy = 0$

(3) $\cos x \cos y\, dx - \sin x \sin y\, dy = 0$

(4) $\frac{1}{y}dx - \frac{x}{y^2}dy = 0$

(5) $(3e^{3x}y - 2x)dx + e^{3x}dy = 0$

(6) $(\mathrm{e}^{\mathrm{x}}\cos y - 2)dx - e^x \sin y\, dy = 0$

(7) $(\tan x + y^2)dx + (\cot y + 2xy)dy = 0$

(8)$(2x-5y)dx+(2y-5x)dy=0$

(9)$\left(3x^2ye^{x^3}-1\right)dx+e^{x^3}dy=0$

(10)$(x+y+3)dx+(x-y-4)dy=0$

(11)$\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$

(12)$(3x^2y-\tan y)dx+(x^3-x\,sec^2y)dy=0$

(13) $\left(\frac{1}{x^2y}+5y\right)dx+\left(\frac{1}{xy^2}+5x\right)dy=0$

(14)$\frac{2x(y-2)}{2+x^2}dx+\ln(2+x^2)\,dy=0$

(15)$(x+\cos y)dx+(\tan y-xsin\,y)dy=0$

(16) $(2ye^{2x}-x^3)dx+e^{2x}dy=0$

■5. REDUCTION TO EXACT D.E. (INTEGRAING FACTOR) ►

The D.E. $-ydx+xdy=0$ is not exact; if we multiply it by $\frac{1}{x^2}$, we get an exact D.E.

$$\frac{-y}{x^2}dx+\frac{1}{x}dy=0$$

By solving the above D.E. the solution is $\frac{y}{x}=c$

This example gives the idea. All we did was multiply a given non exact D.E.

$$P(x,y)dx+Q(x,y)dy=0,$$

by a function $\mu(x,y)\neq 0$ **(integration factor)** the result was

$$\mu P(x,y)dx+\mu Q(x,y)dy=0$$

which is exact equation.

We can obtain the integration factor by several methods such as;

First case

Assume that the integration factor is a function in x only $\mu = \mu(x)$ then the integration factor is

$$\mu(x) = e^{\int \alpha(x)dx} \quad where \quad \alpha(x) = \frac{P_y - Q_x}{Q}$$

Second case

Assume that the integration factor is a function in y only $\mu = \mu(y)$ then the integration factor is

$$\mu(y) = e^{-\int \beta(y)dy} \quad where \quad \beta(y) = \frac{P_y - Q_x}{P}$$

Third case

Assume that the integration factor is $\mu = \mu(x, y)$ and this case hard then the solution is

$$\int_{x_0}^{x} \mu(x,y)P(x,y)dx + \int_{y_0}^{y} \mu(x,y)P(x,y)dy = c$$

EXAMPLE 5.1►

Find the general solution of the D.E

$$(x^2 + y^2 + x)dx + xydy = 0$$

SOLUTION

Let $P = x^2 + y^2 + x$, $Q = xy$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

Then the D.E. is not exact, but we remark that

$$\alpha(x) = \frac{P_y - Q_x}{Q} = \frac{2y - y}{xy} = \frac{1}{x}$$

Then the integration factor is

$$\mu(x) = e^{\int \alpha(x)dx} = e^{\int \frac{1}{x}dx} = x$$

By multiply the given D. E. then we get an exact D.E. which is

$$(x^3 + xy^2 + x^2)dx + x^2ydy = 0,$$

and the solution is

$$3x^4 + 6x^2y^2 + 4x^3 = c$$

EXAMPLE 5.2►

Find the general solution of the D.E

$$(4xy^2 - x^2 + 3y)dx + (2x^2y + x)dy = 0$$

SOLUTION

Let $P = 4xy^2 - x^2 + 3y$, $Q = 2x^2y + x$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

Then the D.E. is not exact, but we remark that

$$\alpha(x) = \frac{P_y - Q_x}{Q} = \frac{4xy + 2}{2x^2y + x} = \frac{2}{x}$$

Then the integration factor is

$$\mu(x) = e^{\int \alpha(x)dx} = x^2$$

By multiply the given D. E. then we get an exact D.E. which is

$$(4x^3y^2 - x^4 + 3x^2y)dx + (2x^4y + x^3)dy = 0,$$

and the solution is

$$5x^4y^2 - x^5 + 5x^3y = c$$

EXAMPLE 5.3►

Find the general solution of the D.E

$$(y\,cos^3x - 1)dx + cos^2x \sin x\, dy = 0$$

SOLUTION

Let $P = y\,cos^3x - 1$, $Q = cos^2x \sin x$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

Then the D.E. is not exact, but we remark that

$$\alpha(x) = \frac{P_y - Q_x}{Q} = \frac{2\cos x\, sin^2 x}{cos^2 x \sin x} = 2\tan x$$

Then the integration factor is

$$\mu(x) = e^{\int \alpha(x)dx} = e^{2\int \tan x dx} = e^{2\ln secx} = sec^2 x$$

By multiply the given D. E. then we get an exact D.E. which is

$$(y\, cosx - sec^2 x)dx + \sin x\, dy = 0,$$

and the solution is

$$y\sin x - \tan x = c$$

EXAMPLE 5.4►

Find the general solution of the D.E

$$(xy^2 + y)dx - xdy = 0$$

SOLUTION

Let $P = xy^2 + y$, $Q = -x$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

Then the D.E. is not exact, but we remark that

$$\beta(y) = \frac{P_y - Q_x}{P} = \frac{2xy + 2}{xy^2 + y} = \frac{2}{y}$$

Then the integration factor is

$$\mu(y) = e^{-\int \beta(y)dy} = \frac{1}{y^2}$$

By multiply the given D. E. then we get an exact D.E. which is

$$\left(x + \frac{1}{y}\right)dx - \left(\frac{x}{y^2}\right)dy = 0,$$

and the solution is

$$\frac{1}{2}x^2 + \frac{x}{y} = c$$

EXAMPLE 5.5►

Find the general solution of the D.E

$$(2\cos 2x + 3y)dx + (3x + \sin 2x + 3xy)\,dy = 0$$

SOLUTION

Let $P = 2\cos 2x + 3y\,,\quad Q = 3x + \sin 2x + 3xy$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

Then the D.E. is not exact, but we remark that

$$\beta(x) = \frac{P_y - Q_x}{P} = \frac{-2\cos 2x - 3y}{2\cos 2x + 3y} = -1$$

Then the integration factor is

$$\mu(y) = e^{-\int \beta(y)dy} = e^{\int dy} = e^{y}$$

By multiply the given D. E. then we get an exact D.E. which is

$$(2\cos 2x + 3y)e^{y}dx + (3x + \sin 2x + 3xy)e^{y}\,dy = 0,$$

and the solution is

$$e^{y}(\sin 2x + 3xy) = c$$

EXAMPLE 5.6►

Find the general solution of the D.E

$$(2xy^4e^{y} + 2xy^3 + y)dx + (x^2y^4e^{y} - x^2y^2 - 3x)dy = 0$$

SOLUTION

Let $P = 2xy^4e^{y} + 2xy^3 + y\,,\quad Q = x^2y^4e^{y} - x^2y^2 - 3x$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

Then the D.E. is not exact, but we remark that

$$\beta(x) = \frac{P_y - Q_x}{P} = \frac{8xy^3e^y + 8xy^2 + 4}{2xy^4e^y + 2xy^3 + y} = \frac{4}{y}$$

Then the integration factor is

$$\mu(y) = e^{-\int \beta(y)dy} = \frac{1}{y^4}$$

By multiply the given D. E. then we get an exact D.E. which is

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + (x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4})dy = 0,$$

and the solution is

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

EXERCISES 5

Find the general solution of the following D.E.

(1) $(1 - xy)dx + (xy - x^2)dy = 0$

(2) $(x^3 - y)dx + (x + x^2y)dy = 0$

(3) $y(y + 1)dx - x(y + 2)dy = 0$

(4) $(3x^2 + y^2)dx + 3xydy = 0$

(5) $(x - y^2)dx + 2xydy = 0$

(6) $ydx + (2x - y^3)dy = 0$

(7) $(x^2 - y^2)dx + 2xydy = 0$

(8) $(3x^2 + 2y)dx + (2x \ln 3x + \frac{3x}{y})dy = 0$

(9) $(2y - 4x^2)dx + xdy = 0$

(10) $(2y - x^3)dx + xdy = 0$